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THE NON-STATIONARY CONTACT PROBLEM FOR ROUGH BODIES TAKING HEAT GENERATION BY FRICTION INTO ACCOUNT†

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The plane contact problem of the theory of elasticity for a rough half-plane when heat is generated due to friction is considered. It is assumed that a constant impressing force is applied with a certain eccentricity to a punch that is parabolic in plan, and that, at the initial instant of time, it begins to execute antiplane motion with a constant velocity. The problem is reduced to a system of integral equations in the contact pressure and the heat fluxes induced in the half-space and the punch, and a numerical algorithm is developed for solving it. Copyright © 1996 Elsevier Science Ltd.

1. FORMULATION OF THE PROBLEM

An elastic parabolic punch, moving uniformly with velocity V in the direction of the z axis (Fig. 1), is impressed by a force P, applied with an eccentricity e, into an elastic homogeneous half-space. The antiplane deformation of the bodies caused by the motion of the punch is assumed to be independent of the deformation plane. It is assumed that in the contact region the forces of friction $\tau_{zy}(x, t)$ are related to the normal stresses $\sigma_y(x, t)$ by Amonton's law $\tau_{zy}(x, t) = f\sigma_y(x, t)$ (f is the coefficient of friction), and there are no shear forces $\tau_{xy}(x, t)$.

The surfaces of the body outside the contact region are thermally insulated, and there is ideal thermal contact in the contact region.

The radius of curvature of the punch base r is sufficiently large, so that when solving the problem of thermoelasticity for the punch it can be replaced by a half-space.

With these assumptions it is required to determine the dimensions of the contact region $A(t)$, the distribution of the contact pressure $p(x, t) \equiv -\sigma_y(x, t)$, the temperatures $T_i(x, y, t)$ and the thermal fluxes $q_i(x, t)$, $(i = 1, 2)$ in the bodies and on their surfaces.

2. REDUCTION OF THE PROBLEM TO A SYSTEM OF INTEGRAL EQUATIONS

The transient temperatures of the surfaces of the bodies can be found from the formula [1]

$$T_i(x,t) = \frac{1}{2\pi\lambda_i}\int_0^t \int_{A(\tau)} q_i(\xi,\tau)\frac{\exp(-\bar{R}_i^2)}{t-\tau}d\xi d\tau, \quad \bar{R}_i = \frac{x-\xi}{2\sqrt{k_i(t-\tau)}} \tag{2.1}$$

$$|x| < \infty, \quad t > 0$$

where k_i, λ_i ($i = 1, 2$) are the thermal diffusivity and the thermal conductivity of the punch and the half-space, respectively.

We will represent the vertical displacements of the punch boundary ($i = 1$) and the half-space ($i = 2$) in the form

$$v_i(x,t) = v_i^e(x,t) + v_i^t(x,t) + v_i^a(x,t), \quad x \in A(t), \quad y = 0, \quad t > 0 \tag{2.2}$$

Here $v_i^e(x, t)$ are the elastic displacements due to the action of the normal stresses $\sigma_y(x, t)$, $v_i^t(x, t)$ are the temperature displacements due to frictional heat generation, and $v_i^a(x, t)$ is the displacement resulting from the micro-roughness in the contact region.

We find the elastic displacements from the solution of the problem of the theory of elasticity in the quasi-stationary formulation [2]

$$v_i^e(x,t) = (-1)^{i+1}\frac{1-\nu_i}{\pi\mu_i}\int_{A(t)} p(\xi,t)\ln|\xi - x|d\xi \quad x \in A(t), \quad t > 0 \tag{2.3}$$

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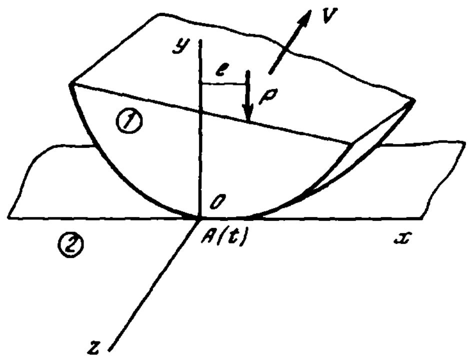

Fig. 1.

where ν_i, μ_i ($i = 1, 2$) are Poisson's ratio and the shear modulus of the materials of the bodies in contact, respectively.

The normal thermal distortions of the surface of an elastic half-space due to the action on the part of its surface $x \in A(t), t > 0, y = 0$ of heat sources of power $q_i(x, t)$ ($i = 1, 2$) are [3]

$$v_i^t(x,t) = (-1)^i \frac{\delta_i k_i}{\pi} \int_0^t \int_{A(t)} q_i(\xi,\tau) \frac{F_1(\tilde{R}_i)}{\sqrt{k_i(t-\tau)}} d\xi d\tau, \quad x \in A(t), \quad t > 0 \tag{2.4}$$

$$\delta_i = \frac{\alpha_i(1+\nu_i)}{\lambda_i}, \quad F_1(\tilde{R}_i) = \frac{2}{\sqrt{\pi}} \frac{\exp(-\tilde{R}_i^2)}{\tilde{R}_i} \int_0^{\tilde{R}_i} \exp(-s^2) ds$$

(α_i ($i = 1, 2$) are the coefficients of linear thermal expansion of the materials of the bodies).

We will assume a linear relationship [4] between $v^a_i(x, t)$ and the contact pressure

$$v_i^a(x,t) = (-1)^{i+1} \beta_i p(x,t), \quad x \in A(t), \quad t > 0 \tag{2.5}$$

where β_i are constants which depend on the degree of roughness of the bodies [4]. Note that relation (2.5) also describes the deformation of a Winkler-type linear coating.

Using relations (2.1) we will satisfy the conditions for non-ideal thermal contact between the bodies taking heat generation into account [6]

$$q_1(x,t) + q_2(x,t) = fVp(x,t), \quad x \in A(t), \quad y = 0, \quad t > 0$$

$$h_0[q_1(x,t) - q_2(x,t)] = T_1(x,t) - T_2(x,t)$$

(h_0 is the contact thermal resistance), and using relations (2.2)–(2.5) we satisfy the condition for mechanical contact

$$v_1(x,t) - v_2(x,t) = \Delta_1(t) + x\Delta_2(t) - g(x), \quad x \in A(t), \quad y = 0, \quad t > 0$$

where $\Delta_1(t)$, $\Delta_2(t)$ are the relative normal displacement and the inclination of the contacting bodies as rigid wholes, respectively, and $g(x) = x^2/(2r)$ is the function describing the punch base.

We put

$$x = A_0 x^*, \quad \xi = A_0 \xi^*, \quad t = \frac{A_0^2}{k_1} t^*, \quad \tau = \frac{A_0^2}{k_1} \tau^*, \quad A(t) = A_0 A^*(t^*)$$

$$T_i(x,t) = \frac{fVP}{\lambda_1} T_i^*(x^*,t^*), \quad q_i(x,t) = \frac{fVP}{A_0} q_i^*(x^*,t^*) \quad (i = 1,2) \tag{2.6}$$

$$p(x,t) = \frac{P}{A_0} p^*(x^*,t^*), \quad A_0 = \frac{3}{4} \frac{\pi(1-\nu_1)\lambda_1}{\alpha_1 fV\mu}, \quad \mu = \frac{2(1-\nu_1^2)\mu_1\mu_2}{(1-\nu_1)\mu_2 + (1-\nu_2)\mu_1}$$

where A_0 is the half-length of the contact region when the boundary surface of the half-space is thermally insulated, and the temperature field of the punch is steady [7].

Using the notation in (2.6) we can write the system of integral equations and equilibrium conditions of the bodies in the form (we omit the asterisks)

$$A_6[q_1(x,t)+q_2(x,t)]+\int_{A(t)}[q_1(\xi,t)+q_2(\xi,t)]\ln|\xi-x|d\xi-\frac{3}{8}\pi\int_0^t\int_{A(\tau)}q_1(\xi,\tau)\frac{F_1(\hat{R}_1)}{\sqrt{t-\tau}}d\xi d\tau-$$

$$-\frac{3}{8}\frac{A_2}{A_3}\pi\int_0^t\int_{A(\tau)}q_2(\xi,\tau)\frac{F_1(\hat{R}_2)}{\sqrt{A_2(t-\tau)}}d\xi d\tau=\Delta_1(t)+x\Delta_2(t)\frac{x^2}{A_4^2}$$

$$A_5[q_1(x,t)-q_2(x,t)]-\int_0^t\int_{A(\tau)}q_1(\xi,\tau)\frac{\exp(-\hat{R}_1^2)}{t-\tau}d\xi d\tau+A_1\int_0^t\int_{A(\tau)}q_2(\xi,\tau)\frac{\exp(-\hat{R}_2^2)}{t-\tau}=0,$$

$$x\in A(t),\quad t>0 \tag{2.7}$$

$$\int_{A(t)}[q_1(\xi,t)+q_2(\xi,t)]d\xi=1,\quad \int_{A(t)}\xi[q_1(\xi,t)+q_2(\xi,t)]d\xi=A_7 \tag{2.8}$$

Here

$$\hat{\Delta}_1(t)=\frac{2r\Delta_1(t)}{A^2(0)},\quad \hat{\Delta}_2(t)=\frac{2rA_0\Delta_2(t)}{A^2(0)},\quad \hat{R}_1=\frac{x-\xi}{2\sqrt{t-\tau}}$$

$$\hat{R}_2=\frac{x-\xi}{2\sqrt{A_2(t-\tau)}},\quad A_1=\frac{\lambda_1}{\lambda_2},\quad A_2=\frac{k_1}{k_2},\quad A_3=\frac{\delta_1}{\delta_2},\quad A_4=\frac{A(0)}{A_0}$$

$$A_5=\frac{2\pi\lambda_1 h_0}{A_0},\quad A_6=\frac{\pi\mu(\beta_1+\beta_2)}{2A_0(1-\nu_1^2)},\quad A_7=\frac{e}{A_0},\quad A^2(0)=\frac{4P(1-\nu_1^2)r}{\pi\mu}$$

($A(0)$ is the half-length of the contact region of the corresponding isothermal Hertz problem [2]).
The following physical inequalities define the contact region

$$p(x,t)\geqslant 0,\quad x\in A(t),\quad t>0$$

$$v_1(x,t)-v_2(x,t)\geqslant \Delta_1(t)+x\Delta_2(t)-g(x),\quad x\,\overline{\in}\, A(t),\quad t>0 \tag{2.9}$$

3. NUMERICAL SOLUTION OF THE SYSTEM OF INTEGRAL EQUATIONS

We divide the time interval $[0, t]$ into l parts of length $\delta t = t/l$: $0 = \tau_0 < \tau_1 < \ldots < \tau_{l-1} < \tau_l = t$. We introduce into the contact region $A(t) = [a(t), b(t)]$ a uniform grid

$$a(\tau)=\xi_0<\xi_1<\ldots<\xi_{n-1}<\xi_n=b(\tau),\quad \xi_i=a(\tau)+i\delta\xi,\quad i=0,1,\ldots,n$$

$$\delta\xi=[b(\tau)-a(\tau)]/n$$

Using a set of piecewise-linear finite functions $\{\varphi_i(\xi)\}^n_{i=0}$ [8] we construct an approximation of the functions $p(\xi, t)$, $q_r(\xi, t)$, $(r = 1, 2)$ in the form

$$p(\xi,t)\cong p^n(\xi,t)=\sum_{i=0}^{n}p(\xi_i,t)\varphi_i(\xi) \tag{3.1}$$

$$q_r(\xi,t)\cong q_r^n(\xi,t)=\sum_{i=0}^{n}q_r(\xi_i,t)\varphi_i(\xi),\quad r=1,2 \tag{3.2}$$

The uniform error of this approximation for functions of the class $C^2([a(t), b(t)]; \mathbb{R}^1)$ is of the order of $\delta\xi^2$ [8].
Using expansion (3.1) for the function

$$J^e(x_k,t_l)=\int_{A(t_l)}p(\xi,t_l)\ln|\xi-x_k|d\xi,\quad x_k=\xi_k,\quad k=0,1,\ldots,n$$

we find its approximate value in the form

$$J_n^e(x_k,t_l)=\sum_{i=0}^{n}M_{ik}^e p_{il},\quad p_{il}\equiv p(\xi_i,t_l)$$

Here

$$M^e_{0k} = \alpha^e_{1k}, \quad M^e_{ik} = \beta^e_{ik} + \alpha^e_{i+1,k}, \quad i = 1,2,\ldots,n-1; \quad M^e_{nk} = \beta^e_{nk}$$

$$\alpha^e_{ik} = \frac{1}{2}\delta\xi \ln|\delta\xi| + \delta\xi[\varkappa_0(\kappa - i + 1) - \varkappa_1(\kappa - i + 1)]$$

$$\beta^e_{ik} = \frac{1}{2}\delta\xi \ln|\delta\xi| + \delta\xi\,\varkappa_1(k - i + 1), \quad i,k = 1,2,\ldots,n$$

$$\varkappa_0(k) = k\ln|k| - (k-1)\ln|k-1| - 1$$

$$\varkappa_1(k) = \frac{1}{2}\left[(1-k^2)\ln|k-1| + k^2\ln|k| - k - \frac{1}{2}\right]$$

Using expansion (3.2) we approximate the function

$$J^l_r(x_k,t_l) = \int_0^{t_l}\int_{A(\tau)} \frac{q_r(\xi,\tau)}{\sqrt{\tilde{k}_r(t_l-\tau)}} F_1\left(\frac{x_k-\xi}{2\sqrt{\tilde{k}_r(t_l-\tau)}}\right) d\xi\, d\tau, \quad r = 1,2$$

$$\tilde{k}_1 = 1, \quad \tilde{k}_2 = A_2,$$

by the sum

$$J^l_{r,n} = \sum_{i=0}^{n}\sum_{j=1}^{l} M^{lr}_{ijkl} q^r_{ij}, \quad q^r_{ij} \equiv q^r_{ij}(\xi_i,\tau_j)$$

$$M^{lr}_{0jkl} = \alpha^{lr}_{1jkl}, \quad M^{lr}_{ijkl} = \beta^{lr}_{ijkl} + \alpha^{lr}_{i+1,jkl}, \quad i = 1,2,\ldots,n-1$$

$$M^{lr}_{njkl} = \beta^{lr}_{njkl}, \quad \alpha^{lr}_{ijkl} = \frac{1}{\delta\xi}(\xi_i c^{0r}_{ijkl} - c^{1r}_{ijkl})$$

$$\beta^{lr}_{ijkl} = \frac{1}{\delta\xi}(c^{1r}_{ijkl} - \xi_{i-1}c^{0r}_{ijkl}), \quad i = 1,2,\ldots,n; \quad r = 1,2$$

$$c^{0r}_{ijkl} = \begin{cases} -\kappa_1[F_4]_1 + \kappa_2[F_4]_2, & j \neq l \\ \kappa_2[F_4]_2, & j = l \end{cases}$$

$$X^r_{qp} = \frac{\hat{x}_p}{2\sqrt{\tilde{k}_r\hat{t}_q}}, \quad \kappa_q = \frac{\tilde{k}_r\hat{t}_q}{\tilde{k}_2}, \quad [F_4]_q = F_4(X^r_{q2}) - F_4(X^r_{q1}), \quad p,r,q = 1,2$$

$$\hat{x}_1 = (k-i)\,\delta x, \quad \hat{x}_2 = (k-i+1)\,\delta x, \quad \hat{t}_1 = (i-j-1/2)\,\delta t, \quad \hat{t}_2 = (l-j+1/2)\,\delta t$$

The influence functions c^{1r}_{ijkl} ($r = 1, 2$) differ from c^{0r}_{ijkl} by the factor $(\tilde{k}_r\hat{t}_q)^{1/2}$ when $[F_4]_q$ is replaced by $[\Psi_4]_q$. The form of the functions $F_4(\cdot)$, $\Psi_4(\cdot)$ is given in [3, 9].

Similarly, using (3.2) we construct an approximation of the function

$$I^r(x_k,t_l) = \int_0^{t_l}\int_{A(\tau)} \frac{q_r(\xi,\tau)}{t_l-\tau}\exp\left(-\frac{x_k-\xi}{2\sqrt{\tilde{k}_r(t_l-\tau)}}\right) d\xi\, d\tau, \quad r = 1,2$$

Taking these approximations into account, we reduce integral equation (2.7) and (2.8) to a system of algebraic equations of order $2n + 2$ in the same number of unknowns q^r_{ik} ($i, k = 1, 2, \ldots, n; r = 1, 2$), $\Delta_1(t_1)$ and $\Delta_2(t_1)$. This system was solved numerically for a number of values of the input parameters $A_1, \ldots, A_7$. The limits of the contact region $a(t)$ and $b(t)$ were found by an iterative method by a direct check of inequalities (2.9).

Knowing the thermal fluxes $q^{(r)}_{ij}$ ($r = 1, 2; i = 1, 2, \ldots, n; j = 1, 2, \ldots, t$), we can determine the contact pressure and the temperature from the formulae

$$p_{il} = q^{(1)}_{il} + q^{(2)}_{il}, \quad i = 1,2,\ldots,n$$

$$T^{(r)}_{kl} = N_r\sum_{j=1}^{l}\sum_{i=1}^{n} q^{(r)}_{ij} d^{(r)}_{ijkl}, \quad N_1 = \frac{1}{2\pi}, \quad N_2 = \frac{A_1}{2\pi}, \quad k = 1,2,\ldots,n$$

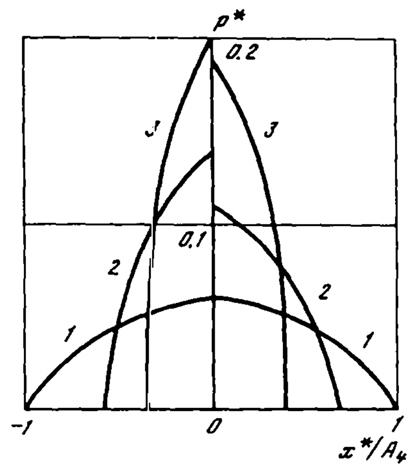

Fig. 2.

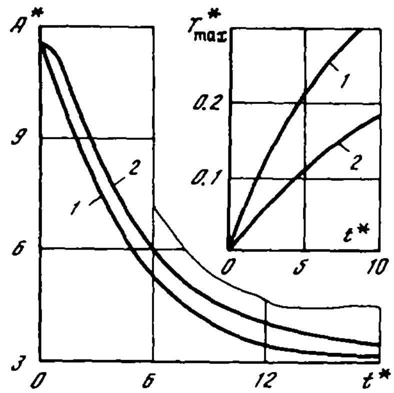

Fig. 3.

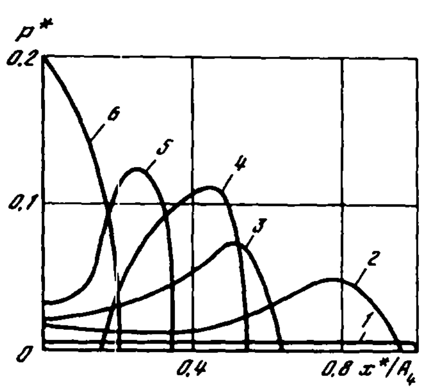

Fig. 4.

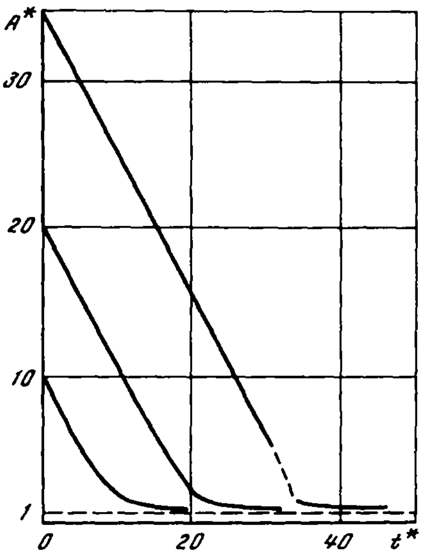

Fig. 5.

In order to reduce the number of parameters $A_1, \ldots, A_7$ we assumed in the calculations that the contact thermal resistance is small ($A_5 = 10^{-3}$) and there is no eccentricity ($A_7 = 0$). In addition, we investigated the heat generation for two cases of friction between the bodies:

1. one of the bodies is thermally insulated, i.e. $A_1 = A_2 = A_3 = 0$ (version 1);
2. the bodies are identical, i.e. $A_1 = A_2 = A_3 = 1$ (version 2).

Hence, the number of input parameters has been reduced to two: A_4, representing the extent to which the surfaces are identical, and A_6, which specifies the microgeometry of the contact.

In Fig. 2 we show the change in the diagram of the dimensionless contact pressure $p^*(x^*, t^*)$ (24), where $A_4 = 10$ and $A_6 = 5$ for three values of the dimensionless time $t^* = 0$, 5 and 10 (curves 1–3, respectively). On the left-hand side in Fig. 2 we show the results of a calculation for version 1, while on the right-hand side we show the results for version 2. The form of the curves indicates that the pressure distribution is close in form to that of the Hertz distribution.

In Fig. 3 we show the change with time of the half-length of the contact section A^* and the maximum value of the temperature in the contact region T_{max}. Curves 1 and 2 correspond to calculations of versions 1 and 2 with $A_4 = 10$ and $A_6 = 5$. It can be seen that when $0 < t^* < 8$ the half-length of the interaction region decreases linearly, reaching a steady value when $t^* \cong 20$. Note also that because of the assumption that the surfaces of the bodies are thermally insulated outside the contact region, the temperature field is not found to reach a steady state.

Another picture in the behaviour of the diagram of the contact pressure is observed when A_4 increases. In Fig. 4 we show $p^*(x^*, t^*)$ as a function of t^* for $A_1 = A_2 = A_3 = 0, A_4 = 35, A_6 = 0$. Curves 1–6 relate to the instants of dimensionless time $t^* = 0, 2.5, 30, 33, 38$ and 40, respectively. It can be seen that as time passes the maximum value of the pressure gradually shifts from the centre to the edge of the contact region. When $t^* \cong 31$ in the central part of the contact area separation of the interacting surfaces occurs. This ring shape of the contact region exists up to $t^* \cong 37$, after which it again becomes simply connected. The pressure becomes equalized, reaching a steady value.

The change of the half-length of the interaction region A^* with time is shown in Fig. 5. Curves 1–3 correspond to $A_4 = 10$, 20 and 35. The dashed curve corresponds to the time interval during which separation of the contact region occurs.

Note that when A_4 is increased further there may be several regions of separation. Thus, for $A_4 \cong 55$ (version 1) there will be two contact regions, while for $A_4 \cong 80$ there will be three.

Hence, we have established that during transient heat generation due to friction there is a minimum value of the parameter A_4 (for the problem considered $A_4 \cong 30$ for version 1 and $A_4 \cong 37$ for version 2), for which the contact region becomes multiconnected. This is due to the fact that, for a fixed load, the slipping velocity V reaches a critical value corresponding to the beginning of thermal instability. The determination of this value is a separate problem.

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